

ON THE EXPANSIONS OF REAL NUMBERS IN TWO MULTIPLICATIVE DEPENDENT BASES

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ABSTRACT. Let $r \geq 2$ and $s \geq 2$ be multiplicatively dependent integers. We establish a lower bound for the sum of the block complexities of the r -ary expansion and of the s -ary expansion of an irrational real number, viewed as infinite words on $\{0, 1, \dots, r-1\}$ and $\{0, 1, \dots, s-1\}$, and we show that this bound is best possible.

1. INTRODUCTION

Throughout this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . Let $b \geq 2$ be an integer. For a real number ξ , write

$$\xi = \lfloor \xi \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor \xi \rfloor + 0.a_1a_2\dots,$$

where each digit a_k is an integer from $\{0, 1, \dots, b-1\}$ and infinitely many digits a_k are not equal to $b-1$. The sequence $\mathbf{a} := (a_k)_{k \geq 1}$ is uniquely determined by the fractional part of ξ . With a slight abuse of notation, we call it the b -ary expansion of ξ and we view it also as the infinite word $\mathbf{a} = a_1a_2\dots$ over the alphabet $\{0, 1, \dots, b-1\}$.

For an infinite word $\mathbf{x} = x_1x_2\dots$ over a finite alphabet and for a positive integer n , set

$$p(n, \mathbf{x}) = \text{Card}\{x_{j+1}\dots x_{j+n} : j \geq 0\}.$$

This notion from combinatorics on words is now commonly used to measure the complexity of the b -ary expansion of a real number ξ . Indeed, for a positive integer n , we denote by $p(n, \xi, b)$ the total number of distinct blocks of n digits in the b -ary

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expansion \mathbf{a} of ξ , that is,

$$p(n, \xi, b) := p(n, \mathbf{a}) = \text{Card}\{a_{j+1} \dots a_{j+n} : j \geq 0\}.$$

Obviously, we have $1 \leq p(n, \xi, b) \leq b^n$, and both inequalities are sharp. If ξ is rational, then its b -ary expansion is ultimately periodic and the numbers $p(n, \xi, b)$, $n \geq 1$, are uniformly bounded by a constant depending only on ξ and b . If ξ is irrational, then, by a classical result of Morse and Hedlund [8], we know that $p(n, \xi, b) \geq n + 1$ for every positive integer n , and this inequality is sharp.

Definition 1.1. A Sturmian word \mathbf{x} is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + 1, \quad \text{for } n \geq 1.$$

A quasi-Sturmian word \mathbf{x} is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + k, \quad \text{for } n \geq n_0,$$

for some positive integers k and n_0 .

The following rather general problem was investigated in [2]. Recall that two positive integers x and y are called *multiplicatively independent* if the only pair of integers (m, n) such that $x^m y^n = 1$ is the pair $(0, 0)$.

Problem 1.2. *Are there irrational real numbers having a ‘simple’ expansion in two multiplicatively independent bases?*

We established in [3] that the complexity function of the r -ary expansion of an irrational real number and that of its s -ary expansion cannot both grow too slowly when r and s are multiplicatively independent positive integers.

Theorem 1.3 ([3]). *Let r and s be multiplicatively independent positive integers. Any irrational real number ξ satisfies*

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) = +\infty.$$

Said differently, ξ cannot have simultaneously a quasi-Sturmian r -ary expansion and a quasi-Sturmian s -ary expansion.

We complement Theorem 1.3 by the following statement addressing expansions of a real number in two multiplicatively dependent bases.

Theorem 1.4. *Let $r, s \geq 2$ be multiplicatively dependent integers and m, ℓ be the smallest positive integers such that $r^m = s^\ell$. Then, there exist uncountably many real numbers ξ satisfying*

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) = m + \ell$$

and every irrational real number ξ satisfies

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) \geq m + \ell.$$

The next result, used in the proof of Theorem 1.4, has its own interest.

Theorem 1.5. *Let $b \geq 2$ be an integer and ρ, σ be positive integers. If σ divides ρ , then every real number whose b^ρ -ary expansion is quasi-Sturmian has a quasi-Sturmian b^σ -ary expansion. Moreover, every real number whose b^ρ -ary and b^σ -ary expansions are both quasi-Sturmian has a quasi-Sturmian b^μ -ary expansion, where μ is the least common multiple of ρ and σ .*

We conclude by an immediate consequence of Theorems 1.3 and 1.4.

Corollary 1.6. *Let $r, s \geq 2$ be distinct integers. No real number can have simultaneously a Sturmian r -ary expansion and a Sturmian s -ary expansion.*

Our paper is organized as follows. Section 2 gathers auxiliary results on Sturmian and quasi-Sturmian words. Theorems 1.4 and 1.5 are established in Section 4.

2. AUXILIARY RESULTS

Here and below, for integers i, j with $i \leq j$, we write x_i^j for the factor $x_i x_{i+1} \dots x_j$ of \mathbf{x} .

We will make use of the following characterisation of quasi-Sturmian words.

Lemma 2.1. *An infinite word \mathbf{x} written over a finite alphabet \mathcal{A} is quasi-Sturmian if and only if there are a finite word W , a Sturmian word \mathbf{s} defined over $\{0, 1\}$ and a morphism ϕ from $\{0, 1\}^*$ into \mathcal{A}^* such that $\phi(01) \neq \phi(10)$ and*

$$\mathbf{x} = W\phi(\mathbf{s}).$$

Proof. See [4]. □

Throughout this paper, for a finite word W and an integer t , we write W^t for the concatenation of t copies of W and W^∞ for the concatenation of infinitely many copies of W . We denote by $|W|$ the length of W , that is, the number of letters composing W . A word U is called periodic if $U = W^t$ for some finite word W and an integer $t \geq 2$. If U is periodic, then the period of U is defined as the length of the shortest word W for which there exists an integer $t \geq 2$ such that $U = W^t$.

Lemma 2.2. *Let U be a finite word. Assume that there exist words U_1, U_2, V, W such that $U = U_1U_2$ and $UU = VU_2U_1W$, with $|U_1| \neq |V|$ and $0 < |V| < |U|$. Then, the word U is periodic.*

Proof. Since V is a prefix of U and W is a suffix of U , we get

$$U = U_1U_2 = VW,$$

thus, $VU_2U_1W = UU = VWVW$. This implies

$$U_2U_1 = WV.$$

If $|U_1| < |V|$, then we can write $V = V'U_1$ for a nonempty word V' , thus $U_2 = WV'$. Therefore,

$$U_1WV' = U_1U_2 = VW = V'U_1W.$$

Our assumption $0 < |V| < |U|$ implies that the word $Z := U_1W$ is nonempty. Since $ZV' = V'Z$, it follows from Theorem 1.5.3 of [1] that $U = ZV'$ is periodic. The proof of the case $|U_1| > |V|$ is similar. \square

Lemma 2.3. *Let \mathcal{A} be a finite set, \mathbf{s} a Sturmian word over $\{0, 1\}$, and ϕ a morphism from $\{0, 1\}^*$ into \mathcal{A}^* satisfying $\phi(01) \neq \phi(10)$. Then there exists an integer n_0 such that, for any factor A of \mathbf{s} of length greater than n_0 , if one can write $\phi(A)$ as $V_1\phi(b_2b_3 \dots b_{m-1})V_2$, where $B = b_1b_2 \dots b_{m-1}b_m$ is a factor of \mathbf{s} , the word V_1 is a nonempty suffix of $\phi(b_1)$, and V_2 is a nonempty prefix of $\phi(b_m)$, then $V_1 = \phi(b_1)$, $V_2 = \phi(b_m)$ and $A = B$.*

Proof. We may assume that 1 is the isolated letter in \mathbf{s} , i.e., that 11 is not a factor of \mathbf{s} . Since \mathbf{s} is balanced, there exists a positive integer k such that 10^t1 is a factor of \mathbf{s} if and only if $t = k$ or $k + 1$.

We first consider the case where $V_1 = \phi(b_1)$. Suppose that $A \neq B$. Then, by deleting the maximal common prefix of A and B , we may assume that A and B have no common prefix. Thus, the prefixes of A and B are 00 and 10 .

If $\phi(00) = \phi(10)V_2$, then $\phi(0) = \phi(1)V_2 = V_2\phi(1)$ and there exist a word U and positive integers s, t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

If $\phi(10) = \phi(0^h)V_2$ for some integer $h \geq 2$ and a nonempty prefix V_2 of $\phi(0)$, then, writing $\phi(0) = V_2V'$, we get $\phi(0) = V_2V' = V'V_2$, thus there exist a word U and positive integers s, t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

If $\phi(10) = \phi(0^h)V_2$ for some integer $h \geq 2$ and a nonempty prefix V_2 of $\phi(1)$, then there exists a positive integer ℓ and a prefix V' of $\phi(0)$ such that $\phi(1) = \phi(0)^\ell V'$. Write $\phi(0) = V'V''$. Then, $\phi(10) = \phi(0)^\ell V' \phi(0) = \phi(0)^{\ell+1} V'$ and we get $V' \phi(0) = \phi(0) V'$. Thus, there exist a word U and positive integers s, t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

Similarly, we show that, if $V_2 = \phi(b_m)$, then $A = B$.

It only remains for us to treat the case where $V_1 \neq \phi(b_1)$ and $V_2 \neq \phi(b_m)$. There exists an integer n_0 such that any factor A of \mathbf{s} of length greater than n_0 contains $10^k 10^{k+1} 10$. It is sufficient to consider the case where $\phi(10^k 10^{k+1} 10) = V_1 \phi(b_2 b_3 \dots b_{m-1}) V_2$, for a factor $b_1 b_2 \dots b_m$ of \mathbf{s} and with V_1 a proper nonempty suffix of $\phi(b_1)$ and V_2 a proper nonempty prefix of $\phi(b_m)$.

If $b_2 b_3 \dots b_{m-1} = 0^{k+1} 10^k 1$, then $b_1 = 1$ and $b_m = 0$. Thus $|V_1| < |\phi(1)|$ and $|V_2| < |\phi(0)|$, which contradicts

$$|V_1| + |V_2| < |\phi(1)| + |\phi(0)| = |\phi(10^k 10^{k+1} 10)| - |\phi(0^{k+1} 10^k 1)|.$$

Therefore, since any subword of \mathbf{s} in which $10^k 10$ and $10^{k+1} 1$ do not occur is a factor of $0^{k+1} 10^k 1$, we deduce that if $\phi(10^k 10^{k+1} 10) = V_1 \phi(b_2 \dots b_{m-1}) V_2$ as above, then $b_2 \dots b_{m-1}$ contains $10^k 10$ or $10^{k+1} 1$.

We distinguish three cases:

Case (i) : $\phi(10^k 10^{k+1} 10) = W_1 \phi(10^k 10) W_2$, where $0 < |W_1| < |\phi(10^k)|$.

Then

$$\phi(10^k 10^k) = W_1 \phi(10^k) W_2', \quad \phi(0^k 10^k 10) = W_1' \phi(0^k 10) W_2,$$

where $|W_2'| = |W_2| - |\phi(0)|$ and $|W_1'| = |W_1|$.

Case (ii) : $\phi(10^k 10^{k+1} 10) = W_1 \phi(10^k 10) W_2$, where $|\phi(10^k)| < |W_1| < |\phi(10^{k+1})|$.

Then

$$\phi(10^k 10^k) = W_1' \phi(0^k 1) W_2', \quad \phi(0^k 100^k 10) = W_1'' \phi(0^k 10) W_2,$$

where $|W_1'| = |W_1| - |\phi(0^k)|$, $|W_2'| = |W_2| + |\phi(0^{k-1})|$ and $|W_1''| = |W_1|$.

Case (iii) : $\phi(10^k 10^{k+1} 10) = W_1 \phi(10^{k+1} 1) W_2$, where $0 < |W_1| < |\phi(10^{k+1})|$.

Then

$$\phi(10^k 10^k) = W_1 \phi(10^k) W_2', \quad \phi(0^k 100^k 10) = W_1' \phi(0^{k+1} 1) W_2,$$

where $|W_2'| = |W_2| - |\phi(0)|$ and $|W_1'| = |W_1|$.

By Lemma 2.2, in each Case (i), (ii), (iii), the factors $\phi(10^k)$ and $\phi(0^k 10)$ are periodic. Denoting by λ_1, λ_2 the periods of $\phi(10^k)$, $\phi(0^k 10)$, we get

$$\lambda_1 \leq \frac{|\phi(10^k)|}{2} = \frac{k|\phi(0)| + |\phi(1)|}{2}, \quad \lambda_2 \leq \frac{|\phi(0^k 10)|}{2} = \frac{(k+1)|\phi(0)| + |\phi(1)|}{2}.$$

Write $\phi(10^k) = U^t$ for a word U with $|U| = \lambda_1$ and integer $t \geq 2$. Then $\phi(1) = U^{t_1} U_1$, $\phi(0^k) = U_2 U^{t_2}$ for some words U_1, U_2 with $U = U_1 U_2$ and some nonnegative integers t_1, t_2 satisfying $t_1 + t_2 = t - 1$. Thus, we get

$$\phi(0^k 1) = U_2 (U_1 U_2)^{t_2} (U_1 U_2)^{t_1} U_1 = (U_2 U_1)^t, \quad |U_2 U_1| = \lambda_1.$$

Since $\phi(0)$ is a prefix of $(U_2 U_1)^t$, we deduce that $\phi(0^k 10) = (U_2 U_1) \cdots (U_2 U_1) U'$ for a prefix U' of $U_2 U_1$. It then follows from [5, Lemma 3 (v)] that $\lambda_1 = \lambda_2$ or

$$|\phi(0^k 10)| < \lambda_1 + \lambda_2 \leq (k + \frac{1}{2})|\phi(0)| + |\phi(1)| < |\phi(0^k 10)|,$$

in which case we have a contradiction. If $\lambda_1 = \lambda_2$, then λ_1 divides $|\phi(0^k 10)|$ and $|\phi(10^k)|$, thus λ_1 divides $|\phi(0)|$ and $|\phi(1)|$. This implies that $\phi(01) = \phi(10) = UU \cdots U$, giving again a contradiction. \square

We end this section with an easy result on the convergents of irrational numbers.

Lemma 2.4. *Let $(\frac{p_k}{q_k})_{k \geq 0}$ be the sequence of convergents of an irrational number $[0; a_1, a_2, \dots]$ in $(0, 1)$ and $d \geq 2$ be an integer. Let c_1, c_2 be integers not both multiple of d . Then, for any positive integer k , we have $c_1 p_k + c_2 q_k \not\equiv 0 \pmod{d}$ or $c_1 p_{k+1} + c_2 q_{k+1} \not\equiv 0 \pmod{d}$.*

Proof. Since

$$\begin{bmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{k+1} \end{bmatrix},$$

we have

$$\begin{bmatrix} c_1 p_k + c_2 q_k & c_1 p_{k+1} + c_2 q_{k+1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{k+1} \end{bmatrix},$$

thus

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} c_1 p_k + c_2 q_k & c_1 p_{k+1} + c_2 q_{k+1} \end{bmatrix} \begin{bmatrix} -a_{k+1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -a_2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -a_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, if $\begin{bmatrix} c_1 p_k + c_2 q_k & c_1 p_{k+1} + c_2 q_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ modulo d , then c_1 and c_2 are multiple of d . \square

3. PROOFS OF THEOREMS 1.4 AND 1.5

We begin with the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $b \geq 2$ be an integer and ρ, σ be positive integers. Assume that $\rho = d\sigma$ for some integer $d \geq 2$. Let ξ be a real number and assume that there are integers a_1, a_2, \dots in $\{0, 1, \dots, b^\rho - 1\}$ and k, n_0 such that

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{a_i}{b^{\rho i}} \quad \text{and} \quad p(n, \xi, b^\rho) = n + k \text{ for } n \geq n_0.$$

Then, by Lemma 2.1, there are a finite word W , a Sturmian word \mathbf{s} defined over $\{0, 1\}$ and a morphism ϕ from $\{0, 1\}^*$ into $\{0, 1, \dots, b^\rho - 1\}^*$ such that $\phi(01) \neq \phi(10)$ and

$$\mathbf{a} = a_1 a_2 \dots = W\phi(\mathbf{s}).$$

Let a be in $\{0, 1, \dots, b^\rho - 1\}$ and consider its representation in base b^σ given by $a = c_1 b^{(d-1)\sigma} + c_2 b^{(d-2)\sigma} + \dots + c_d b^{0 \cdot \sigma}$, where c_1, \dots, c_d are in $\{0, 1, \dots, b^\sigma - 1\}$. Define the function $\phi_{\rho, \sigma}$ on $\{0, 1, \dots, b^\rho - 1\}$ by setting $\phi_{\rho, \sigma}(a) = c_1 c_2 \dots c_d$. It extends to a morphism from $\{0, 1, \dots, b^\rho - 1\}^*$ to $\{0, 1, \dots, b^\sigma - 1\}^*$, which we also denote by $\phi_{\rho, \sigma}$. Then, we have

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{d_i}{b^{\sigma i}}, \quad \text{where } \mathbf{d} = d_1 d_2 \dots = \phi_{\rho, \sigma}(W) (\phi_{\rho, \sigma} \circ \phi)(\mathbf{s}).$$

We deduce from Lemma 2.1 that the b^σ -ary expansion of ξ is quasi-Sturmian. Thus we have established the first assertion of the theorem.

For the second assertion of the theorem, we may assume that ρ and σ are relatively prime (otherwise, we replace b by b^g where g is the greatest common divisor of ρ and σ).

Let ξ be a real number and write

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{a_i}{b^{\rho i}} = \lfloor \xi \rfloor + \sum_{j \geq 1} \frac{b_j}{b^{\sigma j}},$$

where a_1, a_2, \dots are in $\{0, 1, \dots, b^\rho - 1\}$ and b_1, b_2, \dots are in $\{0, 1, \dots, b^\sigma - 1\}$. Assume that $\mathbf{a} = a_1 a_2 \dots$ and $\mathbf{b} = b_1 b_2 \dots$ are both quasi-Sturmian. By Lemma 2.1, there are a finite word W , a Sturmian word \mathbf{s} defined over $\{0, 1\}$ and a morphism ϕ from $\{0, 1\}^*$ into $\{0, 1, \dots, b^\rho - 1\}^*$ such that $\phi(01) \neq \phi(10)$ and

$$\mathbf{a} = a_1 a_2 \dots = W\phi(\mathbf{s}).$$

We claim that $|\phi(0)| =: l_0$ and $|\phi(1)| =: l_1$ are both multiple of σ .

In order to deduce a contradiction, we suppose that σ does not divide at least one of l_0 and l_1 .

Let $\phi_{\rho,1}$ be the morphism $\phi_{\rho,\sigma}$ defined above in the case $\sigma = 1$. For each factor U of \mathbf{s} , let

$$\Lambda(U) := \{0 \leq j \leq \sigma-1 : \phi_{\rho,1}(\mathbf{a}) = V\phi_{\rho,1} \circ \phi(U) \text{ for some } V \text{ with } |V| \equiv j \pmod{\sigma}\}$$

denote the nonempty set of positions modulo σ where $\phi_{\rho,1} \circ \phi(U)$ occurs in $\phi_{\rho,1}(\mathbf{a})$. If U' is a prefix of U , then $\Lambda(U)$ is a subset of $\Lambda(U')$. Consequently, there exists N such that $\Lambda(s_1 \dots s_n) = \Lambda(s_1 \dots s_N)$ for each $n \geq N$.

Let $[0; a_1, a_2, \dots]$ denote the continued fraction expansion of the slope of \mathbf{s} and, for $k \geq 1$, let q_k be the denominator of the convergent $[0; a_1, \dots, a_k]$ to this slope. Define the sequence $(M_k)_{k \geq 0}$ of finite words over $\{0, 1\}$ by

$$M_0 = 0, \quad M_1 = 0^{a_1-1}1, \quad \text{and} \quad M_{k+1} = (M_k)^{a_k} M_{k-1}, \quad (k \geq 1).$$

For $k \geq 1$, the word M_k is a factor of length q_k of \mathbf{s} (see e.g. [7]). Since there are p_k occurrences of the digit 1 in M_k , we get

$$|\phi(M_k)| = l_0(q_k - p_k) + l_1 p_k = (l_1 - l_0)p_k + l_0 q_k.$$

By Lemma 2.4 and the assumption that σ does not divide at least one of l_0 and l_1 , we conclude that at least one of $|\phi(M_k)|$ and $|\phi(M_{k+1})|$ is not a multiple of σ .

Let U be a factor of \mathbf{s} . Then U is a factor of M_k for some integer k . Since $M_k M_k$ is a factor of $M_{k+2} M_{k+1} = (M_{k+1})^{a_{k+2}} M_k (M_k)^{a_{k+1}} M_{k-1}$, which is a factor of \mathbf{s} , there are two positions of $\phi(U)$ which differ by $|\phi(M_k)|$. Thus, there exist two occurrences of $\phi(U)$ in $\phi(\mathbf{s})$ separated by exactly $\rho|\phi(M_k)|$ letters. Replacing k by $k+1$ is necessary, we can assume that $\rho|\phi(M_k)|$ is not a multiple of σ and we deduce that $|\Lambda(U)| \geq 2$ for any factor U of \mathbf{s} .

A finite word U is called right special if U is a prefix of two different factors of \mathbf{s} of the same length. If the initial word $s_1 \dots s_n$ of \mathbf{s} is not a prefix of a right special word, then either $s_{j+1} \dots s_{j+n} \neq s_1 \dots s_n$ for all $j \geq 1$, or \mathbf{s} is periodic. Since a Sturmian word is recurrent and not periodic (see, e.g., [6, page 158]), there are infinitely many prefixes $s_1 \dots s_n$ of \mathbf{s} which are right special. Let $n \geq N$ be such that $s_1 \dots s_n$ is right special. Then, there exists a letter c such that $c \neq s_{n+1}$ and $s_1 \dots s_n c$ is a factor of \mathbf{s} . Thus, we get

$$\Lambda(s_1 \dots s_n s_{n+1}) = \Lambda(s_1 \dots s_n) \supset \Lambda(s_1 \dots s_n c).$$

Choose i, j in $\Lambda(s_1 \dots s_n c)$ with $0 \leq i < j \leq \sigma - 1$. Then we can write

$$\phi_{\rho,1}(\mathbf{a}) = U U_1 \phi_{\rho,1} \circ \phi(s_1 \dots s_n c) U'_1 \dots = U' U_2 \phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1}) U'_2 \dots$$

and

$$\phi_{\rho,1}(\mathbf{a}) = V V_1 \phi_{\rho,1} \circ \phi(s_1 \dots s_n c) V'_1 \dots = V' V_2 \phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1}) V'_2 \dots,$$

for some words $U, U', V, V', U_1, U_2, V_1, V_2, U'_1, U'_2, V'_1, V'_2$ written over $\{0, \dots, b-1\}$ and satisfying

$$|U_1| = |U_2| = i, \quad |V_1| = |V_2| = j, \quad |U| \equiv |U'| \equiv |V| \equiv |V'| \equiv 0 \pmod{\sigma},$$

$$0 \leq |U'_1| = |U'_2| \leq \sigma - 1, \quad 0 \leq |V'_1| = |V'_2| \leq \sigma - 1,$$

and σ divides $i + (n+1)\rho + |U'_1|$ and $j + (n+1)\rho + |V'_1|$. Thus, there exist u_1, u_2, v_1, v_2 in $\{0, 1, \dots, b^\sigma - 1\}$ and words X, Y, A_1, A_2, B_1, B_2 written over $\{0, 1, \dots, b^\sigma - 1\}$ with

$$|X| = \left\lfloor \frac{i + n\rho}{\sigma} \right\rfloor - 1, \quad |Y| = \left\lfloor \frac{j + n\rho}{\sigma} \right\rfloor - 1$$

and

$$A_1 \neq A_2, \quad B_1 \neq B_2, \quad |A_1| = |A_2| < \frac{\rho}{\sigma} + 2, \quad |B_1| = |B_2| < \frac{\rho}{\sigma} + 2,$$

such that

$$\begin{aligned} U_1 \phi_{\rho,1} \circ \phi(s_1 \dots s_n c) U_1' &= \phi_{\sigma,1}(u_1 X A_1), \\ U_2 \phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1}) U_2' &= \phi_{\sigma,1}(u_2 X A_2), \\ V_1 \phi_{\rho,1} \circ \phi(s_1 \dots s_n c) V_1' &= \phi_{\sigma,1}(v_1 Y B_1), \\ V_2 \phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1}) V_2' &= \phi_{\sigma,1}(v_2 Y B_2). \end{aligned}$$

Here, $\phi_{\sigma,1}$ is defined analogously as $\phi_{\rho,1}$. Therefore, $u_1 X A_1$, $u_2 X A_2$ and $v_1 Y B_1$, $v_2 Y B_2$ are all factors of $\phi_{\sigma,1}^{-1}(\phi_{\rho,1}(\phi(\mathbf{s})))$. Denoting by A (resp., by B) the longest common prefix (it could be the empty word) of A_1 and A_2 (resp., of B_1 and B_2), we deduce that XA and YB are both right special.

Let W_0 be the longest common prefix of $\phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1})$ and $\phi_{\rho,1} \circ \phi(s_1 \dots s_n c)$. Then, there exist finite words W_1, W_2, W_1', W_2' over $\{0, \dots, b-1\}$ satisfying $|W_1| = \sigma - i$, $|W_2| = \sigma - j$, $|W_1'| < \sigma$, $|W_2'| < \sigma$, and

$$W_0 = W_1 \phi_{\sigma,1}(XA) W_1' = W_2 \phi_{\sigma,1}(YB) W_2',$$

Thus, we get $|XA| \leq |YB| \leq |XA| + 1$.

Suppose that XA is a suffix of YB . Then, there exists a nonempty finite word W' of length less than σ such that

$$\begin{aligned} W_0 &= W_2 W' \phi_{\sigma,1}(XA) W_1' = W_2 \phi_{\sigma,1}(XA) W_2', & \text{if } |XA| = |YB|, \\ W_0 &= W_1 \phi_{\sigma,1}(XA) W_1' = W_1 W' \phi_{\sigma,1}(XA) W_2', & \text{if } |XA| + 1 = |YB|. \end{aligned}$$

It then follows from Theorem 1.5.2 of [1] that we have $W_0 = W_2(W')^t W'' W_1'$ or $W_1(W')^t W'' W_2'$, respectively, for some integer t and a prefix W'' of W' . Since ρ, σ are fixed and \mathbf{s} is Sturmian, we deduce from Lemma 2.3 of [3] that $(W')^t$ cannot be a factor of $\phi_{\rho,1} \circ \phi(s_1 \dots s_n)$ when n is sufficiently large. This shows that the lengths of XA and YB are bounded independently of n .

Consequently, the right special words XA and YB are not suffixes of each others if n is sufficiently large. Hence, there are arbitrarily large integers m such that $\phi_{\sigma,1}^{-1} \circ \phi_{\rho,1} \circ \phi(\mathbf{s})$ has two distinct right special words of length m . This implies that

$\mathbf{b} = \phi_{\sigma,1}^{-1} \circ \phi_{\rho,1}(\mathbf{a})$ is not quasi-Sturmian, which gives a contradiction. Therefore, we have established that $|\phi(0)|$ and $|\phi(1)|$ are both multiple of σ .

Write

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{c_i}{b^{\rho \sigma i}}, \quad \mathbf{c} = c_1 c_2 \dots = \phi_{\rho \sigma, \rho}^{-1}(\mathbf{a}) = \phi_{\rho \sigma, \rho}^{-1}(W\phi(\mathbf{s})).$$

Put $|W| = h\sigma + d$ for integers $h \geq 0$ and d with $0 \leq d < \sigma$. Let $\phi(0) = X_1 X_2$, $\phi(1) = Y_1 Y_2$, where $|X_1| = |Y_1| = \sigma - d$. Assume that 11 is not a factor of \mathbf{s} . Then there exists a positive integer k such that $10^m 1$ is a factor of \mathbf{s} if and only if $m = k$ or $k + 1$. Thus, we can represent \mathbf{s} as

$$\mathbf{s} = 0^w t_0 t_1 t_2 t_3 \dots, \quad t_0 = 10^k, \quad t_i \in \{10^k, 0\}, \quad 0 \leq w \leq k + 1.$$

It is not difficult to check that $\mathbf{t} := t_0 t_1 t_2 \dots$ is Sturmian. Define ϕ' by

$$\phi'(10^k) = X_2 Y_1 Y_2 (X_1 X_2)^{k-1} X_1, \quad \phi'(0) = X_2 X_1.$$

Then we get

$$\phi(\mathbf{s}) = (X_1 X_2)^w Y_1 Y_2 (X_1 X_2)^{k-1} X_1 \phi'(t_1 t_2 t_3 \dots),$$

thus

$$\mathbf{c} = \phi_{\rho \sigma, \rho}^{-1}(W\phi(\mathbf{s})) = \phi_{\rho \sigma, \rho}^{-1}(W(X_1 X_2)^w Y_1 Y_2 (X_1 X_2)^{k-1} X_1 (\phi_{\rho \sigma, \rho}^{-1} \circ \phi')(t_1 t_2 t_3 \dots)).$$

Since $|\phi(0)|$ and $|\phi(1)|$ are both multiple of σ , the morphism $\phi_{\rho \sigma, \rho}^{-1} \circ \phi'$ is well-defined. We conclude that \mathbf{c} is quasi-Sturmian and the proof of the theorem is complete. \square

Lemma 3.1. *Let $b \geq 2$, $d \geq 2$, ρ, σ be positive integers with $\rho = d\sigma$. Let $x_1 x_2 \dots$ be a quasi-Sturmian word over $\{0, 1, \dots, b^\rho - 1\}$. Then, there exists an integer n_0 such that the real number $\xi = \sum_{k \geq 1} \frac{x_k}{b^{\rho k}}$ satisfies*

$$p(nd, \xi, b^\sigma) \geq (n + 1)d, \quad \text{for } n \geq n_0.$$

Furthermore, if $s_1 s_2 \dots$ is a Sturmian word written over $\{0, 1\}$, then there exists an integer n_0 such that the real number $\xi = \sum_{k \geq 1} \frac{s_k}{b^{\rho k}}$ satisfies

$$p(n, \xi, b^\sigma) = n + d, \quad \text{for } n \geq n_0.$$

Proof. Set $\mathcal{A} := \{0, 1, \dots, b^\rho - 1\}$. There exist a Sturmian word \mathbf{s} written over $\{0, 1\}$, a morphism ϕ from $\{0, 1\}^*$ into \mathcal{A}^* satisfying $\phi(01) \neq \phi(10)$, and a factor W of $\mathbf{x} := x_1 x_2 \dots$ such that $\mathbf{x} = W\phi(\mathbf{s})$. Then, the word

$$\mathbf{y} := \phi_{\rho, \sigma}(\mathbf{x}) = \phi_{\rho, \sigma}(W\phi(\mathbf{s})) = \phi_{\rho, \sigma}(W)(\phi_{\rho, \sigma} \circ \phi)(\mathbf{s})$$

is quasi-Sturmian.

Let n be a positive integer larger than the integer n_0 given by Lemma 2.3 applied to the morphism $\phi_{\rho, \sigma} \circ \phi$. We claim that if $U_1 \phi_{\rho, \sigma}(A_1) V_1 = U_2 \phi_{\rho, \sigma}(A_2) V_2$, where A_1, A_2 are factors of $\phi(\mathbf{s})$ of length n and U_1, U_2 (resp., V_1, V_2) are nonempty suffixes (resp., proper prefixes) of words of the form $\phi_{\rho, \sigma}(a)$ for a in \mathcal{A} , then $U_1 = U_2$, $A_1 = A_2$ and $V_1 = V_2$.

Suppose not. Then we may assume that there exist A_1, A_2 and U, V such that

$$\phi_{\rho, \sigma}(A_1) V = U \phi_{\rho, \sigma}(A_2).$$

Thus there exist a_1, a_2 in \mathcal{A} , a factor A of $\phi(\mathbf{s})$ of length n , and a factor A' of $\phi(\mathbf{s})$ of length $n - 1$ such that $\phi_{\rho, \sigma}(A) = W_1 \phi_{\rho, \sigma}(A') W_2$, where W_1 (resp., W_2) is a nonempty proper suffix (resp., prefix) of $\phi_{\rho, \sigma}(a_1)$ (resp., of $\phi_{\rho, \sigma}(a_2)$). Consequently, there exist b, b', c, c' in $\{0, 1\}$ and factors B, B' of \mathbf{s} such that $A = U \phi(B) V$, $a_1 A' a_2 = U' \phi(B') V'$, where U (resp., U') is a nonempty suffix of $\phi(b)$ (resp., $\phi(b')$) and V (resp., V') is a nonempty prefix of $\phi(c)$ (resp., $\phi(c')$). Then $A' = U'' \phi(B') V''$ for words U'', V'' such that $U' = a_1 U''$, $V' = V'' a_2$. Therefore, we get

$$\phi_{\rho, \sigma}(A) = \phi_{\rho, \sigma}(U)(\phi_{\rho, \sigma} \circ \phi)(B) \phi_{\rho, \sigma}(V) = W_1 \phi_{\rho, \sigma}(U'')(\phi_{\rho, \sigma} \circ \phi)(B') \phi_{\rho, \sigma}(V'') W_2.$$

We deduce from Lemma 2.3 that $\phi_{\rho, \sigma}(U) = W_1 \phi_{\rho, \sigma}(U'')$, $\phi_{\rho, \sigma}(V) = \phi_{\rho, \sigma}(V'') W_2$ and $B = B'$. This is a contradiction to the fact that W_1 (resp., W_2) is a nonempty proper suffix (resp., prefix) of $\phi_{\rho, \sigma}(a_1)$ (resp., of $\phi_{\rho, \sigma}(a_2)$). Hence, the representation of $X = U \phi_{\rho, \sigma}(A) V$ is unique.

If $\phi(\mathbf{s})$ is written over an alphabet of three letters or more, then

$$p(n - 1, \phi(\mathbf{s})) \geq (n - 1) + 2 = n + 1,$$

which implies that the number of factors X of $(\phi_{\rho, \sigma} \circ \phi)(\mathbf{s})$ of length nd is at least equal to $(n + 1)d$. If $\phi(\mathbf{s})$ is written over an alphabet of two letters, say over the alphabet $\mathcal{A} = \{a, b\}$, then we can put $\phi_{\rho, \sigma}(a) = ZX$ and $\phi_{\rho, \sigma}(b) = ZY$, where Z is the longest common prefix of $\phi_{\rho, \sigma}(a), \phi_{\rho, \sigma}(b)$ and the first letters of X, Y are

different. If $|V| > |Z|$, then for each right special factor A of \mathbf{s} there are two distinct factors $\phi_{\rho,\sigma}(A)V_1, \phi_{\rho,\sigma}(A)V_2$ in $\phi(\mathbf{s})$. If $|V| \leq |Z|$, then $|U| \geq |X| = |Y|$, thus for each left special factor B of \mathbf{s} there are two factors $U_1\phi_{\rho,\sigma}(B), U_2\phi_{\rho,\sigma}(B)$ in $\phi(\mathbf{s})$. For each $c = 0, \dots, d-1$, the number of factors $X = U\phi_{\rho,\sigma}(A)V$ of $(\phi_{\rho,\sigma} \circ \phi)(\mathbf{s})$ of length nd with $|A| = n-1$ and $|U| = d - |V| = c$ is at least equal to $p(n-1, \phi(\mathbf{s})) + 1$. Therefore, we get

$$p(nd, \xi, b^\sigma) \geq p(nd, (\phi_{\rho,\sigma} \circ \phi)(\mathbf{s})) \geq (n+1)d.$$

Since the function $m \mapsto p(m, \xi, b^\sigma)$ is strictly increasing, this implies the first assertion of the theorem.

For the second assertion, let $\mathbf{s} = s_1s_2\dots$ be a Sturmian word written over the subset $\{0, 1\}$ of $\{0, 1, \dots, b^\rho - 1\}$ and define

$$\xi = \sum_{i \geq 1} \frac{s_i}{b^{\rho i}}.$$

Since $\phi_{\rho,\sigma}(0) = 0^d$ and $\phi_{\rho,\sigma}(1) = 0^{d-1}1$, for $n \geq 1$, any factor of length dn of $\phi_{\rho,\sigma}(\mathbf{s})$ is a suffix of $\phi_{\rho,\sigma}(A)0^k$, where A is a factor of length n in \mathbf{s} and $0 \leq k \leq d-1$. Since 0^{d-1} is a prefix of $\phi_{\rho,\sigma}(A)0^k$, the number of suffixes of $\phi_{\rho,\sigma}(A)0^k$ of length nd is $d(n+1)$, thus

$$p(dn, \xi, b^\sigma) = d(n+1) = dn + d.$$

Since the function $m \mapsto p(m, \xi, b^\sigma)$ is strictly increasing, this completes the proof of the theorem. \square

Proof of Theorem 1.4. Suppose that the two bases $r \geq 2$ and $s \geq 2$ are multiplicatively dependent and let m, ℓ be the coprime positive integers satisfying $r^m = s^\ell$. Then, there exists a positive integer b such that $r = b^\ell$ and $s = b^m$.

Let $\mathbf{s} = s_1s_2\dots$ be a Sturmian word over the subset $\{0, 1\}$ of $\{0, 1, \dots, b^{m\ell} - 1\}$ and define

$$\xi = \sum_{i \geq 1} \frac{s_i}{b^{m\ell i}}.$$

By the second assertion of Lemma 3.1, there exists an integer n_0 such that

$$p(n, \xi, b^\ell) = n + m \quad \text{and} \quad p(n, \xi, b^m) = n + \ell, \quad \text{for } n \geq n_0.$$

Thus,

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) = m + \ell.$$

This proves the first assertion of the theorem.

For the second assertion of the theorem, it is sufficient to consider a real number ξ whose b^ℓ -ary and b^m -ary expansions are both quasi-Sturmian. By Theorem 1.5, the $b^{\ell m}$ -ary expansion of ξ is also quasi-Sturmian and we deduce from the first assertion of Lemma 3.1 that there exists an integer n_0 such that

$$p(mn, \xi, b^\ell) \geq m(n+1) \quad \text{and} \quad p(\ell n, \xi, b^m) \geq \ell(n+1), \quad \text{for } n \geq n_0.$$

Therefore,

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) \geq m + \ell.$$

This completes the proof of the theorem. \square

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